4. Suppose that X, Y are independent exponential random variables with parameter λ.

(3 marks) (a) Find $P(X < 2Y)$.

(4 marks) (b) For $a > 0$, find $P(X > a | X < 2Y)$. What is the conditional distribution (give the name and any parameter(s)) of X given that $X < 2Y$?

(3 marks) (c) Which of the following is a correct formula for $E(X^2 | X > 1)$?
$E[(X + 1)^2]$, $(EX + 1)^2$, $EX^2 + 1$.
Give a brief reason for your answer.

4(a) $P(X<2Y)$

$$= \int_0^\infty \int_0^{2y} \lambda e^{-\lambda x} \lambda e^{-\lambda y}\, dx\, dy$$

$$= \int \int \lambda^2 e^{-\lambda y} e^{-\lambda x}\, dx\, dy$$

$$= \int \lambda^2 e^{-\lambda y} \left(\int_0^{2y} e^{-\lambda x}\, dx \right) dy$$

(A) $\int_0^{2y} e^{-\lambda x}\, dx$

$$= \frac{e^{-\lambda x}}{-\lambda} \Bigg|_{x=0}^{x=2y}$$

$$= \frac{e^{-\lambda(2y)}}{-\lambda} + \frac{e^{-\lambda(0)}}{+\lambda}$$

$$= \frac{1}{\lambda} - \frac{e^{-\lambda 2y}}{\lambda} = \boxed{\left(\frac{1-e^{-2\lambda y}}{\lambda}\right)}$$

$P(X<2y)$

$$= \int \lambda^2 e^{-\lambda y} \boxed{\left(\frac{1-e^{-2\lambda y}}{\lambda}\right)} dy$$

$$= \int \lambda e^{-\lambda y}(1-e^{-2\lambda y})\, dy$$

$$= \lambda \int e^{-\lambda y}(1-e^{-2\lambda y})\, dy$$

$$= \lambda \int e^{-\lambda y} - e^{-2\lambda y} e^{-\lambda y}\, dy$$

$$\lambda \left[\int e^{-\lambda y} dy - \int e^{-3\lambda y} dy \right]$$

$$= \lambda \left[\frac{e^{-\lambda y}}{-\lambda} + \left(\frac{e^{-3\lambda y}}{+3\lambda} \right) \right] \Bigg|_{y=0}^{y=\infty}$$

$$= \left[-e^{-\lambda y} + \frac{e^{-3\lambda y}}{3} \right] \Bigg|_{y=0}^{y=\infty}$$

$$= 0 - \left(e^{0} + \frac{1}{3} \right)$$

$$= 1 - \frac{1}{3} = \boxed{\frac{2}{3}}$$

$P(A|B)P(B) = P(A\cap B)$

4(b) $P(X>a \mid X<2Y)$

$= \dfrac{P(A\cap B)}{P(B)}$

$= \dfrac{P(X>a, X<2Y)}{P(X<2Y)}$

$= \dfrac{P(a<X<2Y)}{P(X<2Y)}$

$P(a<X<2Y)$

$= \boxed{\int_0^\infty \int_a^{2Y} \lambda e^{-\lambda x} \lambda e^{-\lambda y}\, dx\, dy}$

$= \int_0^\infty \int_a^{2y} (\lambda^2 e^{-\lambda y}) e^{-\lambda x}\, dx\, dy$

$= \int_0^\infty \lambda^2 e^{-\lambda y} \boxed{\int_a^{2y} e^{-\lambda x}\, dx}\, dy$

(A) $\int_a^{2y} e^{-\lambda x}\, dx$

$= \dfrac{e^{-\lambda x}}{-\lambda}\Big|_{x=a}^{x=2y}$

$= \dfrac{e^{-\lambda(2y)}}{-\lambda} + \left(\dfrac{e^{-\lambda(a)}}{+\lambda}\right)$

$= \dfrac{e^{-2y\lambda}}{-\lambda} + \dfrac{e^{-\lambda a}}{\lambda}$

$= \left(\dfrac{e^{-\lambda a} - e^{-2y\lambda}}{\lambda}\right)$ ✓

$P(a<X<2y) =$

$\int_0^\infty \lambda^2 e^{-\lambda y}\left(\dfrac{e^{-\lambda a} - e^{-2y\lambda}}{\lambda}\right) dy$

$= \int_0^\infty \lambda e^{-\lambda y}\left(e^{-\lambda a} - e^{-2y\lambda}\right) dy$

$= \lambda \int_0^\infty e^{-\lambda y - \lambda a} - e^{-\lambda y - 2y\lambda}\, dy$

$= \lambda \int_0^\infty e^{-\lambda a} \cdot e^{-\lambda y} - e^{-3y\lambda}\, dy$

$= \lambda \left[\int_0^\infty e^{-\lambda a} e^{-\lambda y}\, dy - \int e^{-3y\lambda}\, dy\right]$

(B) $e^{-\lambda a} \int_0^\infty e^{-\lambda y}\, dy$

$= e^{-\lambda a}\left(\dfrac{e^{-\lambda y}}{-\lambda}\Big|_0^\infty\right) = e^{-\lambda a}\left(0 + \dfrac{1}{\lambda}\right) = \dfrac{e^{-\lambda a}}{\lambda}$

$$\int_0^\infty e^{-3y\lambda}\,dy$$

$$= \left.\frac{e^{-3y\lambda}}{-3\lambda}\right|_0^\infty$$

$$= 0 + \frac{1}{+3\lambda}$$

$$= \frac{1}{3\lambda}$$

$\therefore P(a<X<2y) =$

$$\lambda\left(\frac{e^{-\lambda a}}{\lambda} - \frac{1}{3\lambda}\right)$$

$$= \boxed{e^{-\lambda a} - \frac{1}{3}}$$

$$\frac{P(a<X<2Y)}{P(X<2Y)} = \left(e^{-\lambda a} - \frac{1}{3}\right)\left(\frac{3}{2}\right)$$

$$= \boxed{\frac{3}{2}e^{-\lambda a} - \frac{1}{2}}$$

4(c) $\mathbb{E}[X^2 | X > 1]$

Due to memoryless property of exponential distribution

$\mathbb{E}[X^2 | X > 1] = \mathbb{E}[X^2] + 1$